Fractional quantum mechanics

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A path integral approach to quantum physics has been developed. Fractional path integrals over the paths of the Lévy flights are defined. It is shown that if the fractality of the Brownian trajectories leads to standard quantum and statistical mechanics, then the fractality of the Lévy paths leads to fractional quantum mechanics and fractional statistical mechanics. The fractional quantum and statistical mechanics have been developed via our fractional path integral approach. A fractional generalization of the Schrödinger equation has been found. A relationship between the energy and the momentum of the nonrelativistic quantum-mechanical particle has been established. The equation for the fractional plane wave function has been obtained. We have derived a free particle quantum-mechanical kernel using Fox's *H* function. A fractional generalization of the Heisenberg uncertainty relation has been established. Fractional statistical mechanics has been developed via the path integral approach. A fractional generalization of the motion equation for the density matrix has been found. The density matrix of a free particle has been expressed in terms of the Fox's *H* function. We also discuss the relationships between fractional and the well-known Feynman path integral approaches to quantum and statistical mechanics.

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I. INTRODUCTION

The term ''fractal'' was introduced into scientists' lexicon by Mandelbrot [1]. Historically, the first example of a fractional physical object was Brownian motion, whose trajectories (paths) are nondifferentiable, self-similar curves that have a fractal dimension that is different from its topological dimension [1,2]. In quantum physics the first successful attempt to apply the fractality concept was the Feynman path integral approach to quantum mechanics. Feynman and Hibbs [3] reformulated the nonrelativistic quantum mechanics as a path integral over Brownian paths. Thus the Feynman-Hibbs fractional background leads to standard (nonfractional) quantum mechanics.

We develop an extension of a fractality concept in quantum physics. That is, we construct a fractional path integral and formulate the fractional quantum mechanics [4] as a path integral over the paths of the Lévy flights.

The Lévy stochastic process is a natural generalization of the Brownian motion or the Wiener stochastic process [5,6]. The foundation for this generalization is the theory of stable probability distributions developed by Lévy [7]. The most fundamental property of the Lévy distributions is the stability in respect to addition, in accordance with the generalized central limit theorem. Thus, from the probability theory point of view, the stable probability law is a generalization of the well-known Gaussian law. The Lévy processes are characterized by the Lévy index α, $0<\alpha\leqslant 2$. At $\alpha=2$ we have the Gaussian process or the process of the Brownian motion. Let us note that the Lévy process is widely used to model a variety of processes, such as turbulence [8], chaotic dynamics [9], plasma physics [10], financial dynamics [11], biology, and physiology [12].

As is well known, in the Gaussian case the path integral approach to quantum mechanics allows one to reproduce the Schrödinger equation for the wave function. In the general case we derive the fractional generalization of the Schrödinger equation [see Eq. (28)]. The fractional generalization of the Schrödinger equation includes the derivative of order α instead of the second ($\alpha=2$) derivative in the standard Schrödinger equation. This is one of the reasons for the term ''fractional quantum mechanics'' (FQM).

The paper is organized as follows. In Sec. II we describe two fractals: (i) a trajectory of the Brownian motion, and (ii) a trajectory of the Lévy flight. In Sec. III we define the fractional path integrals in the coordinate and phase space representations. We develop the FQM via a path integral. It is shown in what way the FQM includes the standard one. We derive the free particle fractional quantum-mechanical propagator using Fox's *H* function. The fractional dispersion relation between the energy and the momentum of the nonrelativistic fractional quantum mechanical particle is established.

In Sec. IV the fractional generalization of the Schrödinger equation in terms of the quantum Riesz fractional derivative is obtained. The fractional Hamilton operator is defined, and its hermiticity is proven.

As a physical application of the developed fractional quantum mechanics, a free particle quantum dynamics is studied in Sec. V. We introduce the Lévy wave packet, which is a fractional generalization of the well-known Gaussian wave packet. Quantum-mechanical probability densities in space and momentum representations are derived. The fractional uncertainty relation is established. This uncertainty relation can be considered as a fractional generalization of the Heisenberg uncertainty relation.

In Sec. VI we develop the fractional statistical mechanics (FSM) by means of the fractional path integral approach. The main point is go from imaginary time (in the framework of the quantum-mechanical fractional path integral consideration) to ''inverse temperature'' $it\rightarrow\hbar\beta$, where $\beta=1/k_BT$,

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k_B is Boltzmann's constant, $\hbar$ is Planck's constant, and T is the temperature. We have found an equation for the partition function of the fractional statistical system. The fractional density matrix for a free particle is expressed in analytical form in terms of the Fox's H function. We have derived the new fractional differential equation [see Eq. (63)] for the fractional density matrix. In the conclusion, we discuss the relationships between the fractional approach and the well-known Feynman path integral approach to quantum and statistical mechanics.

II. FRACTALS

The relation between fractals and quantum (or statistical) mechanics is easily observed in the framework of the Feynman path integral formulation [3]. The background of the Feynman approach to quantum mechanics is a path integral over the Brownian paths. The Brownian motion was historically the first example of the fractal in physics. Brownian paths are nondifferentiable, self-similar curves whose fractal dimension is different from its topological dimension. Let us explain the fractal dimension with two examples of fractals: (i) the Brownian path, and (ii) the trajectory of the Lévy flight.

(i) A mathematical model of the Brownian motion is the Wiener stochastic process $x(t)$ [5]. The probability density $p_W(xt|x_0t_0)$, that a stochastic process $x(t)$ will be found at x at time t under the condition that it starts at $t=t_0$ from $x(t_0)=x_0$, satisfies the diffusion equation

$$\frac{\partial p_W(xt|x_0t_0)}{\partial t}=\frac{\sigma}{2}\nabla^2 p_W(xt|x_0t_0),$$

$$p_W(xt|x_0t_0)=\delta(x-x_0), \quad \nabla\equiv\frac{\partial}{\partial x},$$

the solution of which has the form

$$p_W(xt|x_0t_0)\equiv p_W(x-x_0;t-t_0) =\frac{1}{\sqrt{2\pi\sigma(t-t_0)}}\exp\left\{-\frac{(x-x_0)^2}{2\sigma(t-t_0)}\right\}, \quad (1)$$

where σ is the diffusion coefficient, and $t>t_0$.

Equation (1) implies that

$$(x-x_0)^2\propto\sigma(t-t_0). \quad (2)$$

This scaling relation between a length increment of the Wiener process $\Delta x=x-x_0$ and a time increment $\Delta t=t-t_0$ allows one to find the fractal dimension of the Brownian path. Let us consider the length of the diffusion path between two given space-time points. We divide the given time interval T into N slices, such as $T=N\Delta t$. Then the space length of the diffusion path is

$$L=N\Delta x=\frac{T}{\Delta t}\Delta x=\sigma T(\Delta x)^{-1}, \quad (3)$$

where the scaling relation [Eq. (2)] was taken into account. The fractal dimension tells us about the length of the path when space resolution goes to zero, $\Delta x\to 0$. The fractional dimension d_{fractal} may be introduced by [1,2]

$$L\propto(\Delta x)^{1-d_{\text{fractal}}},$$

where $\Delta x\to 0$. Letting $\Delta x\to 0$ in Eq. (3), and comparing with the definition of the fractal dimension d_{fractal}, yields

$$d_{\text{fractal}}^{(\text{Brownian})}=2. \quad (4)$$

Thus the fractal dimension of the Brownian path is 2.

(ii) Another example of a fractal is the random process of the Lévy "flight" (or the Lévy motion). As discussed in Sec. I, the Lévy motion is a so-called α-stable random process, and may be considered as a generalization of the Brownian motion. The α-stable distribution is formed under the influence of the sum of a large number of independent random factors. The probability density $p_L(xt|x_0t_0)$ of the Lévy α-stable distribution has the form

$$p_L(xt|x_0t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ik(x-x_0)}\exp\{-\sigma_\alpha|k|^\alpha(t-t_0)\}, \quad (5)$$

where α is the Lévy index $0<\alpha\leqslant 2$, and σ_α is the generalized diffusion coefficient with the "physical" dimension $[\sigma_\alpha]=\text{cm}^\alpha\,\text{sec}^{-1}$. The α-stable distribution with $0<\alpha<2$ possesses finite moments of order μ, $\mu<\alpha$, but infinite moments for higher order. Note that the Gaussian probability distribution is also a stable one $(\alpha=2)$, and it possesses moments of all orders.

We will further study a fractional quantum and statistical mechanics, and it seems reasonable to suggest that there exists moments of first order or physical averages (for example, an average momentum or space coordinate of quantum particle; see Secs. V and VI). The requirement for the first moment's existence gives the restriction, $1<\alpha\leqslant 2$.

The α-stable Lévy distribution defined by Eq. (5) satisfies the fractional diffusion equation

$$\frac{\partial p_L(xt|x_0t_0)}{\partial t}=\sigma_\alpha\nabla^\alpha p_L(xt|x_0t_0), \quad \nabla^\alpha\equiv\frac{\partial^\alpha}{\partial x^\alpha},$$
$$p_L(xt|x_0t_0)=\delta(x-x_0), \quad (6)$$

where ∇^α is the fractional Riesz derivative defined through its Fourier transform [13,14]:

$$\nabla^\alpha p(x,t)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ikx}|k|^\alpha\overline{p}(k,t). \quad (7)$$

Here $p(x,t)$ and $\overline{p}(k,t)$ are related to each other by the Fourier transforms

$$p(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ikx}\overline{p}(k,t),$$

$$\overline{p}(k,t)=\int_{-\infty}^{\infty}dx\, e^{-ikx}p(x,t).$$

Equation (5) implies that

$$(x-x_0)\propto(\sigma_\alpha(t-t_0))^{1/\alpha},\quad 1<\alpha\leqslant 2. \tag{8}$$

This scaling relation between a length increment of the Lévy process $\Delta x=x-x_0$, and a time increment $\Delta t=t-t_0$, allows one to find the fractal dimension of a trajectory of a Lévy path. Let us consider the length of the Lévy path between two given space-time points. Dividing the given time interval T into N slices, such as $T=N\Delta t$, and taking into account the scaling relation [Eq. (8)], we have

$$L=N\Delta x=\frac{T}{\Delta t}\Delta x=DT(\Delta x)^{1-\alpha}.$$

Letting $\Delta x\to 0$, and comparing with the definition of the fractal dimension $d_{\rm fractal}$ [1,2], yields

$$d_{\rm fractal}^{(\text{Lévy})}=\alpha,\quad 1<\alpha\leqslant 2. \tag{9}$$

Thus the fractal dimension of the considered Lévy path is α.

III. FRACTIONAL PATH INTEGRAL

If a particle at an initial time t_a starts from a point x_a and goes to a final point x_b at time t_b, we will say simply that the particle goes from a to b, and its trajectory (path) $x(t)$ will have the property that $x(t_a)=x_a$ and $x(t_b)=x_b$. In quantum mechanics, then, we will have a quantum-mechanical amplitude, often called a kernel, which we may write $K_F(x_bt_b|x_at_a)$, which we use to get from the point a to the point b. This will be the sum over all the trajectories that go between the end points, and of a contribution from each. If we have a quantum particle moving in the potential $V(x)$ then the quantum-mechanical amplitude $K_F(x_bt_b|x_at_a)$ may be written as [3]

$$K_F(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}_{\rm Feynman}x(\tau)\times\exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b}d\tau\, V(x(\tau))\right\}, \tag{10}$$

where $V(x(\tau))$ is the potential energy as a functional of a particle path $x(\tau)$, and the Feynman path integral measure is defined as

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}_{\rm Feynman}x(\tau)\cdots=\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1\ldots dx_{N-1}\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{-N/2}\times\prod_{j=1}^{N}\exp\left\{\frac{im}{2\hbar\varepsilon}(x_j-x_{j-1})^2\right\}\cdots, \tag{11}$$

here m is the mass of the quantum-mechanical particle, $\hbar$ is the Planck's constant, $x_0=x_a$, $x_N=x_b$, and $\varepsilon=(t_b-t_a)/N$. The Feynman path integral measure is generated by the process of the Brownian motion. Indeed, Eq. (11) implies

$$(x_j-x_{j-1})\propto\left(\frac{\hbar}{m}\right)^{1/2}(\Delta t)^{1/2}.$$

This is the typical relation between the space displacement and the time scale for the Brownian path. This scaling relation between a length increment (x_j-x_{j-1}) and a time increment Δt implies that the fractal dimension of the Feynman's path is $d_{\rm fractal}^{(\rm Feynman)}=2$. As is well known, the definition given by Eq. (11) leads to standard quantum mechanics. We conclude that the Feynman-Hibbs fractional background leads to standard (nonfractional) quantum mechanics [3].

We propose the fractional quantum mechanics based on the fractional path integral

$$K_L(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x(\tau)\exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b}d\tau\, V(x(\tau))\right\}, \tag{12}$$

where $V[x(\tau)]$ is the potential energy as a functional of the Lévy particle path, and the fractional path integral measure is defined as

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x(\tau)\cdots=\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1\ldots dx_{N-1}\hbar^{-N}\times\left(\frac{iD_\alpha\varepsilon}{\hbar}\right)^{-N/\alpha}\prod_{j=1}^{N}L_\alpha\times\left\{\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\varepsilon}\right)^{1/\alpha}|x_j-x_{j-1}|\right\}\cdots, \tag{13}$$

where D_α is the generalized ''fractional quantum diffusion coefficient,'' the physical dimension of which is $[D_\alpha]={\rm erg}^{1-\alpha}\,{\rm cm}^\alpha\,{\rm sec}^{-\alpha}$, $\hbar$ denotes Planck's constant, $x_0=x_a$, $x_N=x_b$, $\varepsilon=(t_b-t_a)/N$, and the Lévy distribution function L_α is expressed in terms of Fox's H function [15–17]:

$$\hbar^{-1}\left(\frac{D_\alpha t}{\hbar}\right)^{-1/\alpha}L_\alpha\left\{\frac{1}{\hbar}\left(\frac{\hbar}{D_\alpha t}\right)^{1/\alpha}|x|\right\}=\frac{1}{\alpha|x|}H_{2,2}^{1,1}\left[\frac{1}{\hbar}\left(\frac{\hbar}{D_\alpha t}\right)^{1/\alpha}|x|\,\middle|\,{}^{(1,1/\alpha),(1,\frac{1}{2})}_{(1,1),(1,\frac{1}{2})}\right]. \tag{14}$$

Here α is the Lévy index and, as mentioned in Sec. II, we consider the case when $1<\alpha\leqslant 2$.

The functional measure defined by Eq. (13) is generated by the Lévy flights stochastic process. We find from Eq. (13) that the scaling relation between a length increment (x_j-x_{j-1}) and a time increment Δt has a fractional form

$$|x_j-x_{j-1}|\propto(\hbar^{\alpha-1}D_\alpha)^{1/\alpha}(\Delta t)^{1/\alpha}.$$

This scaling relation implies that the fractal dimension of the Lévy path is $d_{\rm fractal}^{(\text{Lévy})}=\alpha$. So, in the general case, a $1<\alpha<2$ Lévy fractional background leads to fractional quantum mechanics. Equations (12)–(14) define the new fractional quantum mechanics via the fractional path integral.

As a physical application of the developed fractional path integral approach let us calculate the free particle kernel $K_L^{(0)}(x_bt_b|x_at_a)$, and compare it with the Feynman free particle kernel $K_F^{(0)}(x_bt_b|x_at_a)$. For the free particle $V(x)=0$, and Eqs. (12) and (13) yields

$$K_L^{(0)}(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x(\tau)\times 1$$
$$=\hbar^{-1}\left(\frac{iD_\alpha(t_b-t_a)}{\hbar}\right)^{-1/\alpha}$$
$$\times L_\alpha\left\{\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha(t_b-t_a)}\right)^{1/\alpha}|x_b-x_a|\right\}. \quad (15)$$

It is known that at $\alpha=2$ the Lévy distribution is transformed to a Gaussian, and the Lévy flight process is transformed to the process of Brownian motion. Equation (15), in accordance with the definition given by Eq. (14) and the properties of the Fox's function $H_{2,2}^{1,1}$ at $\alpha=2$ (see Refs. [16], [17]) is transformed to a Feynman free particle kernel [see Eq. (3-3) of Ref. [3]]

$$K_F^{(0)}(x_bt_b|x_at_a)=\left(\frac{2\pi i\hbar(t_b-t_a)}{m}\right)^{-1/2}\exp\left\{\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)}\right\}. \quad (16)$$

Thus the general fractional [Eq. (15)] includes, as a particular, a Gaussian case at $\alpha=2$, the Feynman propagator.

In terms of a Fourier integral (momentum representation), the fractional kernel $K_L^{(0)}(x_bt_b|x_at_a)$ is written as

$$K_L^{(0)}(x_bt_b|x_at_a)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,\exp\left\{i\frac{p(x_b-x_a)}{\hbar}-i\frac{D_\alpha(t_b-t_a)|p|^\alpha}{\hbar}\right\}, \quad (17)$$

while the Eq. (16) in the momentum representation has the form

$$K_F^{(0)}(x_bt_b|x_at_a)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,\exp\left\{i\frac{p(x_b-x_a)}{\hbar}-i\frac{p^2(t_b-t_a)|}{2m\hbar}\right\}. \quad (18)$$

We see from Eq. (17) that the energy E_p of the fractional quantum mechanical particle with momentum p is given by

$$E_p=D_\alpha|p|^\alpha. \quad (19)$$

This is a dispersion relation for the nonrelativistic fractional quantum-mechanical particle. The comparison of the Eqs. (17) and (18) allows one to conclude that at $\alpha=2$ we should put $D_2=1/2m$. Then Eq. (19) is transformed to the standard nonrelativistic equation $E_p=p^2/2m$.

Using Eq. (17), we can define the fractional functional measure in the phase space representation by

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b}Dx(\tau)\int Dp(\tau)\cdots$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1\ldots dx_{N-1}\frac{1}{(2\pi\hbar)^N}$$
$$\times\int_{-\infty}^{\infty}dp_1\ldots dp_N\exp\left\{i\frac{p_1(x_1-x_a)}{\hbar}-i\frac{D_\alpha\varepsilon|p_1|^\alpha}{\hbar}\right\}$$
$$\times\cdots\times\exp\left\{i\frac{p_N(x_b-x_{N-1})}{\hbar}-i\frac{D_\alpha\varepsilon|p_N|^\alpha}{\hbar}\right\}\cdots, \quad (20)$$

here $\varepsilon=(t_b-t_a)/N$. Then the kernel $K_L(x_bt_b|x_at_a)$, defined by Eq. (12), can be written as

$$K_L(x_bt_b|x_at_a)=\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1\ldots dx_{N-1}\frac{1}{(2\pi\hbar)^N}$$
$$\times\int_{-\infty}^{\infty}dp_1\ldots dp_N\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}p_j(x_j-x_{j-1})\right\}$$
$$\times\exp\left\{-\frac{i}{\hbar}D_\alpha\varepsilon\sum_{j=1}^{N}|p_j|^\alpha-\frac{i}{\hbar}\varepsilon\sum_{j=1}^{N}V(x_j)\right\}.$$

In the continuum limit $N\to\infty$ and $\varepsilon\to 0$, we have

$$K_L(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}Dx(\tau)\int Dp(\tau)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}d\tau[p(\tau)\dot{x}(\tau)-H_\alpha(p(\tau),x(\tau))]\right\}, \quad (21)$$

where the phase space path integral $\int_{x(t_a)=x_a}^{x(t_b)=x_b}Dx(\tau)\int Dp(\tau)\ldots$ is given by Eq. (20), $\dot{x}$ denotes the time derivative, H_α is the fractional Hamiltonian

$$H_\alpha(p,x)=D_\alpha|p|^\alpha+V(x) \quad (22)$$

with the replacements $p\to p(\tau)$ and $x\to x(\tau)$, and $\{p(\tau),x(\tau)\}$ is the particle trajectory in phase space. We will discuss the hermiticity property of the fractional Hamiltonian H_α in Sec. IV.

The exponential in Eq. (21) can be written as $\exp\{(i/\hbar)S_\alpha(p,x)\}$ if we introduce the fractional canonical action for the trajectory $\{p(t),x(t)\}$ in phase space:

$$S_\alpha(p,x)=\int_{t_a}^{t_b}d\tau(p(\tau)\dot{x}(\tau))-H_\alpha(p(\tau),x(\tau)). \quad (23)$$

Since the coordinates x_0 and x_N in definition (20) are fixed at their initial and final points $x_0=x_a$ and $x_N=x_b$, all possible trajectories in Eq. (23) satisfy the boundary conditions $x(t_b)=x_b$ and $x(t_a)=x_a$. We see that the definition given by Eq. (20) includes one more p_j integral than x_j integral. Indeed, while x_0 and x_N are held fixed and the x_j integrals are done for $j=1,\ldots,N-1$, each increment x_j

$-x_{j-1}$ is accompanied by one p_j integral for $j=1,...,N$. The above observed asymmetry is a consequence of the particular boundary condition. That is, the end points are fixed in position (coordinate) space. There exists the possibility of proceeding in a conjugate way, keeping the initial p_a and final p_b momenta fixed. The associated kernel can be derived going through the same steps as before, but working in the momentum representation (see, for example, Ref. [18]).

Taking into account Eq. (17), it is easy to check directly on the consistency condition

$$K_L^{(0)}(x_bt_b|x_at_a)=\int_{-\infty}^{\infty}dx'K_L^{(0)}(x_bt_b|x't')K_L^{(0)}(x't'|x_at_a).$$

This is a special case of the general fractional quantum-mechanical rule: amplitudes for events occurring in succession in time multiply

$$K_L(x_bt_b|x_at_a)=\int_{-\infty}^{\infty}dx'K_L(x_bt_b|x't')K_L(x't'|x_at_a). \tag{24}$$

IV. FRACTIONAL SCHRÖDINGER EQUATION

The kernel $K_L(x_bt_b|x_at_a)$, which is defined by Eq. (13), describes the evolution of the fractional quantum-mechanical system

$$\psi_f(x_b,t_b)=\int_{-\infty}^{\infty}dx_aK_L(x_bt_b|x_at_a)\psi_i(x_a,t_a), \tag{25}$$

where $\psi_i(x_a,t_a)$ is the fractional wave function of the initial (at $t=t_a$) state, and $\psi_f(x_b,t_b)$ is the fractional wave function of the final (at $t=t_b$) state.

In order to obtain the differential equation for the fractional wave function $\psi(x,t)$, we apply Eq. (25) in the special case that the time t_b differs only by an infinitesimal interval ε from t_a:

$$\psi(x,t+\varepsilon)=\int_{-\infty}^{\infty}dy\,K_L(x,t+\varepsilon|y,t)\psi(y,t).$$

Using Feynman's approximation $\int_t^{t+\tau}d\tau\,V(x(\tau))\simeq\varepsilon V[(x+y)/2]$ and the definition given by Eq. (13), we have

$$\psi(x,t+\varepsilon)=\int_{-\infty}^{\infty}dy\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\exp\left\{i\frac{p(y-x)}{\hbar}-i\frac{D_\alpha\varepsilon|p|^\alpha}{\hbar}-\frac{i}{\hbar}\varepsilon V\left(\frac{x+y}{2}\right)\right\}\psi(y,t).$$

We may expand the left- and right-hand sides in power series:

$$\psi(x,t)+\varepsilon\frac{\partial\psi(x,t)}{\partial t}=\int_{-\infty}^{\infty}dy\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,e^{i[py-x)/\hbar]}\times\left(1-i\frac{D_\alpha\varepsilon|p|^\alpha}{\hbar}\right)\times\left[1-\frac{i}{\hbar}\varepsilon V\left(\frac{x+y}{2}\right)\right]\psi(y,t). \tag{26}$$

Then, taking into account the definitions of the Fourier transforms,

$$\psi(x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,e^{i(px/\hbar)}\varphi(p,t),$$

$$\varphi(p,t)=\int_{-\infty}^{\infty}dp\,e^{-i(px/\hbar)}\psi(x,t),$$

and introducing the quantum Riesz fractional derivative $(\hbar\nabla)^\alpha$

$$(\hbar\nabla)^\alpha\psi(x,t)=-\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\;e^{i(px/\hbar)}|p|^\alpha\varphi(p,t), \tag{27}$$

we obtain, from Eq. (26),

$$\psi(x,t)+\varepsilon\frac{\partial\psi(x,t)}{\partial t}=\psi(x,t)+i\frac{D_\alpha\varepsilon}{\hbar}(\hbar\nabla)^\alpha\psi(x,t)-\frac{i}{\hbar}\varepsilon V(x)\psi(x,t).$$

This will be true to order ε if $\psi(x,t)$ satisfies the differential equation

$$i\hbar\frac{\partial\psi}{\partial t}=-D_\alpha(\hbar\nabla)^\alpha\psi+V(x)\psi. \tag{28}$$

This is the fractional Schrödinger equation for a fractional quantum particle moving in one dimension.

Equation (28) may be rewritten in operator form, namely

$$i\hbar\frac{\partial\psi}{\partial t}=H_\alpha\psi, \tag{29}$$

where H_α is the fractional Hamiltonian operator:

$$H_\alpha=-D_\alpha(\hbar\nabla)^\alpha+V(x). \tag{30}$$

Using definition (27) one may rewrite the fractional Hamiltonian H_α in the form given by Eq. (23).

The Hamiltonian H_α is the Hermitian operator in space, with a scalar product

$$(\phi,\chi)=\int_{-\infty}^{\infty}dx\;\phi^*(x,t)\chi(x,t).$$

To prove the hermiticity of H_α, let us note that in accordance with the definition of the quantum Riesz fractional derivative given by Eq. (27) there exists the integration-by parts formula

$$(\phi,(\hbar\nabla)^{\alpha}\chi)=((\hbar\nabla)^{\alpha}\phi,\chi). \tag{31}$$

The average energy of fractional quantum system, with Hamiltonian H_α, is

$$E_\alpha=\int_{-\infty}^{\infty}dx\,\psi^*(x,t)H_\alpha\psi(x,t). \tag{32}$$

Taking into account Eq. (31) we have

$$E_\alpha=\int_{-\infty}^{\infty}dx\,\psi^*(x,t)H_\alpha\psi(x,t)$$
$$=\int_{-\infty}^{\infty}dx(H_\alpha^+\psi(x,t))^*\psi(x,t)=E_\alpha^*,$$

and, as a physical consequence, the energy of a system is real. Thus the fractional Hamiltonian H_α defined by Eq. (30) is the Hermitian or self-adjoint operator

$$(H_\alpha^+\phi,\chi)=(\phi,H_\alpha\chi).$$

Since the kernel $K_L(x_bt_b|x_at_a)$, thought of as a function of variables x_b and t_b, is a special wave function (for a particle which starts at x_a and t_a), we see that K_L must also satisfy a fractional Schrödinger equation. Thus, for the quantum system described by the fractional Hamiltonian [Eq. (30)], we have

$$i\hbar\frac{\partial}{\partial t_b}K_L(x_bt_b|x_at_a)=-D_\alpha(\hbar\nabla_b)^\alpha K_L(x_bt_b|x_at_a)$$
$$+V(x_b)K_L(x_bt_b|x_at_a),\quad t_b>t_a,$$

where the low index "b" means that the quantum fractional derivative acts on the variable x_b.

V. FREE PARTICLE. FRACTIONAL UNCERTAINTY RELATION

As a first physical application of the developed FQM and the fractional Schrödinger equation (28), let us consider a free particle. The fractional Schrödinger equation for a free particle has the fractional plane wave solution

$$\psi(x,t)=C\exp\left\{i\frac{px}{\hbar}-i\frac{D_\alpha|p|^\alpha t}{\hbar}\right\}, \tag{33}$$

where C is a normalization constant. In the special Gaussian case ($\alpha=2$ and $D_2=1/2m$), Eq. (33) gives a plane wave of the standard quantum mechanics. Localized states are obtained by a superposition of plane waves:

$$\psi_L(x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,\varphi(p)\exp\left\{i\frac{px}{\hbar}-i\frac{D_\alpha|p|^\alpha t}{\hbar}\right\}. \tag{34}$$

Here, $\varphi(p)$ is the "weight" function. We will study Eq. (34) for a one-dimensional fractional Lévy wave packet,

$$\psi_L(x,t)=\frac{A_\nu}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{2\hbar^\nu}\right\}$$
$$\times\exp\left\{i\frac{px}{\hbar}-i\frac{D_\alpha|p|^\alpha t}{\hbar}\right\}, \tag{35}$$

with the "weight" function

$$\varphi(p)=A_\nu\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{2\hbar^\nu}\right\},\quad p_0>0\quad \nu\leqslant\alpha,$$

where A_ν is a constant, l is a space scale, and α is the Lévy index $1<\alpha\leqslant 2$.

In the following we will be interested in the probability density $\rho(x,t)$ that a particle occupies a position x, and the probability density $w(p,t)$ that a particle has particular values p of the momentum. The wave function $\psi_L(x,t)$, defined by Eq. (35), gives the probability density $\rho(x,t)$:

$$\rho(x,t)=|\psi_L(x,t)|^2$$
$$=\frac{A_\nu^2}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dp_1\,dp_2$$
$$\times\exp\left\{-\frac{|p_1-p_0|^\nu l^\nu}{2\hbar^\nu}\right\}$$
$$\times\exp\left\{-\frac{|p_2-p_0|^\nu l^\nu}{2\hbar^\nu}\right\}$$
$$\times\exp\left\{i\frac{(p_1-p_2)x}{\hbar}-i\frac{D_\alpha(|p_1|^\alpha-|p_2|^\alpha)t}{\hbar}\right\}. \tag{36}$$

Now, we can fix the factor A_ν such that $\int dx\rho(x,t)=\int dx|\psi_L(x,t)|^2=1$, with the result

$$A_\nu=\left(\frac{\pi\nu l}{\Gamma\left(\frac{1}{\nu}\right)}\right)^{1/2}, \tag{37}$$

where $\Gamma(1/\nu)$ is the γ function.[1] The relationship between the probability densities $\rho(x,t)$ and $w(p,t)$ may be derived from the relationship between fractional wave functions in the space $\psi_L(x,t)$ and momentum $\phi(p,t)$ representations,

$$\psi_L(x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\,\exp\left\{i\frac{px}{\hbar}\right\}\phi(p,t), \tag{38}$$

where we have

$$\phi(p,t)=\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{2\hbar^\nu}\right\}\exp\left\{-i\frac{D_\alpha|p|^\alpha t}{\hbar}\right\}. \tag{39}$$

Note that $\phi(p,t)$ satisfies the fractional free particle Schrödinger equation in the momentum representation

[1]The γ function $\Gamma(z)$ has the familiar integral representation $\Gamma(z)=\int_0^\infty dt\,t^{z-1}e^{-t}$, $\mathrm{Re}\,z>0$.

$$i\hbar\frac{\partial\phi(p,t)}{\partial t}=D_\alpha|p|^\alpha\phi(p,t),$$

$$\phi(p,0)=\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{2\hbar^\nu}\right\}.$$

One then obtains

$$\int_{-\infty}^{\infty}dx|\psi_L(x,t)|^2=\frac{A_\nu^2}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,dp'\times\exp\left\{i\frac{(p-p')x}{\hbar}\right\}\phi(p,t)\phi^*(p,t)=\frac{A_\nu^2}{(2\pi\hbar)}\int_{-\infty}^{\infty}dp|\phi(p,t)|^2=1, \quad (40)$$

because of

$$\frac{1}{(2\pi\hbar)}\int_{-\infty}^{\infty}dx\exp\left\{i\frac{(p-p')x}{\hbar}\right\}=\delta(p-p').$$

Equation (40) suggests, for the probability density in momentum space, the following definition:

$$w(p,t)=\frac{A_\nu^2}{2\pi\hbar}|\phi(p,t)|^2. \quad (41)$$

Thus, for the momentum probability density $w(p,t)$, we have

$$w(p,t)\equiv w(p)=\frac{\nu l}{2\hbar\Gamma\left(\frac{1}{\nu}\right)}\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{\hbar^\nu}\right\}. \quad (42)$$

This is time independent, since we are considering a free particle.

In coordinate space the probability of finding a particle at the position x in the ''box'' dx is given by $\rho(x,t)dx$. Correspondingly, the probability of finding a particle with momentum p in dp is represented by $w(p,t)dp$.

We are also interested in the average values and the mean-μ deviations of position and momentum for the present probability densities defined by Eqs. (36) and (42). The expectation value of the space position can be calculated as

$$\langle x\rangle=\int_{-\infty}^{\infty}dx\,x\rho(x,t)=\frac{A_\nu^2}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dx\,x\int_{-\infty}^{\infty}dp\,dp'\times\exp\left\{i\frac{(p-p')x}{\hbar}\right\}\phi(p,t)\phi^*(p',t). \quad (43)$$

Making the substitution

$$x\rightarrow\frac{\hbar}{i}\frac{\partial}{\partial p},$$

we will have

$$\langle x\rangle=\frac{A_\nu^2}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,dp'\times\left(\frac{\hbar}{i}\frac{\partial}{\partial p}\exp\left\{i\frac{(p-p')x}{\hbar}\right\}\right)\phi(p,t)\phi^*(p',t).$$

Integrating by parts gives

$$\langle x\rangle=-\frac{A_\nu^2}{(2\pi\hbar)}\frac{\hbar}{i}\int_{-\infty}^{\infty}dp\left(\frac{l^\nu}{\hbar^\nu}\frac{\partial}{\partial p}|p-p_0|^\nu-i\frac{D_\alpha t}{\hbar}\frac{\partial}{\partial p}|p|^\alpha\right)\exp\left\{-\frac{|p-p_0|^\nu l^\nu}{\hbar^\nu}\right\}.$$

It is easy to check that the first term in the brackets vanishes, and we find that the position expectation value is

$$\langle x\rangle=\alpha D_\alpha t p_0{}^{\alpha-1}. \quad (44)$$

Using the dispersion relation given by Eq. (19), we may rewrite $\langle x\rangle$ as

$$\langle x\rangle=\left.\frac{\partial E_p}{\partial p}\right|_{p=p_0}t=v_0t. \quad (45)$$

Here, $v_0=(\partial E_p/\partial p)|_{p=p_0}$ is the group velocity of the wave packet. We see that the maximum of the Lévy wave packet [Eq. (35)] moves with the group velocity v_0 like a classical particle.

The mean-μ deviations ($\mu<\nu$) of space position $\langle|\Delta x|^\mu\rangle$ is defined by

$$\langle|\Delta x|^\mu\rangle=\langle|x-\langle x\rangle|^\mu\rangle=\int_{-\infty}^{\infty}dx|x-\langle x\rangle|^\mu\rho(x,t)=\frac{A_\nu^2}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}dx|x-\langle x\rangle|^\mu\times\int_{-\infty}^{\infty}dp\,dp'\exp\left\{i\frac{(p-p')x}{\hbar}\right\}\times\phi(p,t)\phi^*(p',t).$$

This equation can be rewritten as

$$\langle|\Delta x|^\mu\rangle=\frac{l^\mu}{2}\mathcal{N}(\alpha,\mu,\nu;\tau,\eta_0), \quad (46)$$

where we introduce the following notations:

$$\mathcal{N}(\alpha,\mu,\nu;\tau,\eta_0)=\frac{2^{1/\nu}\nu}{4\pi\Gamma\left(\frac{1}{\nu}\right)}\int_{-\infty}^{\infty}d\zeta|\zeta|^\mu\int_{-\infty}^{\infty}d\eta\times\int_{-\infty}^{\infty}d\eta'\exp\{i(\eta-\eta')(\zeta+\alpha\tau\eta_0^{\alpha-1})\}\times\exp\{-i\tau(|\eta|^\alpha-|\eta'|^\alpha)-|\eta-\eta_0|^\nu-|\eta'-\eta_0|^\nu\} \quad (47)$$

and

$$\eta_0 = \frac{p_0 l}{2^{1/\nu}\hbar}, \quad \tau = \frac{D_\alpha t}{\hbar}\left(\frac{2^{1/\nu}\hbar}{l}\right)^\alpha$$

So, for the μ-root of the mean-μ deviation of position (space position uncertainty for the Lévy wave packet), we find

$$\langle|\Delta x|^\mu\rangle^{1/\mu} = \frac{l}{2^{1/\mu}}\mathcal{N}^{1/\mu}(\alpha,\mu,\nu;\tau,\eta_0). \tag{48}$$

Further, with Eq. (42), the expectation value of the momentum is calculated as

$$\langle p\rangle = \int_{-\infty}^{\infty} dp\, p w(p) = \int_{-\infty}^{\infty} dp (p-p_0) w(p) + \int_{-\infty}^{\infty} dp\, p_0 w(p). \tag{49}$$

The first integral vanishes, since $w(p)$ is an even function of $(p-p_0)$, and the momentum expectation value is

$$\langle p\rangle = p_0. \tag{50}$$

The mean-μ deviation of the momentum is

$$\langle|\Delta p|^\mu\rangle = \int_{-\infty}^{\infty} dp|p-\langle p\rangle|^\mu w(p) = \left(\frac{\hbar}{l}\right)^\mu \frac{\Gamma\left(\frac{\mu+1}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)}. \tag{51}$$

Then the momentum uncertainty (the μ-root of the mean-μ deviation of momentum) is

$$\langle|\Delta p|^\mu\rangle^{1/\mu} = \frac{\hbar}{l}\left(\frac{\Gamma\left(\frac{\mu+1}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\right)^{1/\mu}. \tag{52}$$

Together with Eq. (48), this leads to

$$\langle|\Delta x|^\mu\rangle^{1/\mu}\langle|\Delta p|^\mu\rangle^{1/\mu} = \frac{\hbar}{2^{1/\mu}}\left(\frac{\Gamma\left(\frac{\mu+1}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\right)^{1/\mu}\mathcal{N}^{1/\mu}(\alpha,\mu,\nu;\tau,\eta_0),$$
$$\mu<\nu\leq\alpha, \tag{53}$$

where $\mathcal{N}(\alpha,\mu,\nu,\tau,\eta_0)$ is given by Eq. (47).

This relation implies that a spatially extended Lévy (or fractional) wave packet corresponds to a narrow momentum spectrum, whereas a sharp Lévy wave packet corresponds to a broad momentum spectrum.

Since $\mathcal{N}(\alpha,\mu,\nu;\tau,\eta_0)>1$ and $\Gamma(\mu+1/\nu)/\Gamma(1/\nu)\approx 1/\nu$, Eq. (53) becomes

$$\langle|\Delta x|^\mu\rangle^{1/\mu}\langle|\Delta p|^\mu\rangle^{1/\mu} > \frac{\hbar}{(2\alpha)^{1/\mu}}, \quad \mu<\alpha, \quad 1<\alpha\leq 2, \tag{54}$$

with $\nu=\alpha$.

Note that for the special case $\alpha=2$ we can set $\mu=\alpha=2$. Thus, for the standard quantum mechanics, ($\alpha=2$) with the definition of the uncertainty as the square-root of the mean-square deviation, Eq. (54) was established by Heisenberg [19] (see, for instance, Ref. [20]). The uncertainty relation given by Eq. (54) can be considered as fractional generalization of the well known Heisenberg uncertainty relations. Thus Eqs. (12)–(15), (21)–(24), (28), (30), and (54) are the basic equations for the new FQM.

VI. FRACTIONAL STATISTICAL MECHANICS

In order to develop the fractional statistical mechanics (FSM), let us go in the previous quantum-mechanical consideration from imaginary time to ''inverse temperature'' $\beta = 1/k_B T$, where k_B is Boltzmann's constant and T is the temperature, $it\to\hbar\beta$. In the framework of the traditional functional approach to the statistical mechanics, we have the functional over the Wiener measure [3,18,21],

$$\rho(x,\beta|x_0) = \int_{x(0)=x_0}^{x(\beta)=x}\mathcal{D}_{\rm Wiener}\, x(u)\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} du V(x(u))\right\} \tag{55}$$

where $\rho(x,\beta|x_0)$ is the density matrix of the statistical system in the external field $V(x)$, and the Wiener measure [6] generated by the process of the Brownian motion is given by

$$\int_{x(0)=x_0}^{x(\beta)=x}\mathcal{D}_{\rm Wiener}\, x(u)\cdots = \lim_{N\to\infty}\int dx_1\ldots dx_{N-1}\left(\frac{2\pi\hbar\varsigma}{m}\right)^{-N/2} \times\prod_{j=1}^{N}\exp\left\{-\frac{m}{2\hbar\varsigma}(x_j-x_{j-1})^2\right\}\cdots, \tag{56}$$

here $\varsigma=\hbar\beta/N$ and $x_N=x$.

The FSM deals with Lévy or fractional density matrix $\rho_L(x,\beta|x_0)$, which is defined by

$$\rho_L(x,\beta|x_0) = \int_{x(0)=x_0}^{x(\beta)=x}\mathcal{D}_{\rm L\acute{e}vy}x(u)\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} du\, V(x(u))\right\} \tag{57}$$

where we introduce the new fractional functional measure (we will call this measure by the Lévy functional measure) by

$$\int_{x(0)=x_0}^{x(\beta)=x}\mathcal{D}_{\rm L\acute{e}vy}x(u)\cdots = \lim_{N\to\infty}\int dx_1\ldots dx_{N-1} \times(\hbar^{\alpha-1}D_\alpha\varsigma)^{-N/\alpha} \times\prod_{j=1}^{N} L_\alpha\left\{\frac{|x_j-x_{j-1}|}{(\hbar^{\alpha-1}D_\alpha\varsigma)^{1/\alpha}}\right\}\cdots, \tag{58}$$

here $\varsigma = \hbar\beta/N$, $x_N = x$, and the Lévy function L_α is given by Eq. (14). Equations (57) and (58) define the fractional quantum statistics via new Lévy path integral.

The partition function Z or free energy F, $Z = e^{-\beta F}$ is expressed as a trace of the density matrix $\rho_L(x,\beta|x_0)$:

$$Z = e^{-\beta F} = \int dx\, \rho_L(x,\beta|x) = \int dx \int_{x(0)=x(\beta)=x} \mathcal{D}_{\text{Lévy}} x(u) \times \exp\left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} du V(x(u)) \right\}.$$

With the definition (20) the equation for the partition function becomes

$$Z = e^{-\beta F} = \int dx \int_{x(0)=x(\beta)=x} Dx(\tau) \int Dp(\tau) \times \exp\left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} du \{ -ip(u)\dot{x}(u) + H_\alpha(p(u),x(u)) \} \right\}, \tag{59}$$

where the fractional Hamiltonian $H_\alpha(p,x)$ has form of Eq. (22), and $p(u)$ and $x(u)$ may be considered as paths running along on ''imaginary time axis,'' $u = it$. The exponential expression of Eq. (59) is very similar to the fractional canonical action [Eq. (23)]. Since it governs the fractional quantum-statistical path integrals, it may be called the fractional quantum-statistical action or fractional Euclidean action, indicated (following Ref. [18]) by the superscript (e),

$$S_\alpha^{(e)}(p,x) = \int_0^{\hbar\beta} du \{ -ip(u)\dot{x}(u) + H_\alpha(p(u),x(u)) \}.$$

The parameter u is not the true time in any sense. It is just a parameter in an expression for the density matrix (see, for instance, Ref. [3]). Let us call u the ''time,'' leaving the quotation marks to remind us that it is not real time (although u does have the dimension of time). Likewise $x(u)$ will be called the ''coordinate'' and $p(u)$ the ''momentum.'' Then Eq. (57) may be interpreted in the following way.

Consider all possible paths by which the system can travel between the initial $x(0)$ and final $x(\beta)$ configurations in the ''time'' $\hbar\beta$. The fractional density matrix ρ_L is a path integral over all possible paths, the contribution from a particular path being the ''time'' integral of the canonical action [considered as the functional of the paths $p(u)$ and $x(u)$ in the phase space] divided by $\hbar$. The partition function is derived by integrating over only those paths for which initial $x(0)$ and final $x(\beta)$ configurations are the same, and after that we integrate over all possible initial (or final) configurations.

The fractional density matrix $\rho_L^{(0)}(x,\beta|x_0)$ of a free particle $(V=0)$ can be written as

$$\rho_L^{(0)}(x,\beta|x_0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \exp\left\{ i\frac{p(x-x_0)}{\hbar} - \beta D_\alpha |p|^\alpha \right\} = \frac{1}{\alpha|x-x_0|} H_{2,2}^{1,1}\left[\frac{|x-x_0|}{\hbar(D_\alpha\beta)^{1/\alpha}} \middle| \begin{matrix} (1,1/\alpha),(1,\frac{1}{2}) \\ (1,1),(1,\frac{1}{2}) \end{matrix} \right], \tag{60}$$

where $H_{2,2}^{1,1}$ is Fox's H function (see Refs. [15–17]). For a linear system of space scale Ω the trace of Eq. (60) leads to

$$Z = e^{-\beta F} = \int_\Omega dx\, \rho_L(x,\beta|x) = \frac{\Omega}{2\pi\hbar} \int_{-\infty}^{\infty} dp \exp\{ -\beta D_\alpha |p|^\alpha \} = \frac{\Omega}{2\pi\hbar} \frac{1}{(\beta D_\alpha)^{1/\alpha}} \Gamma\left(\frac{1}{\alpha}\right).$$

When $\alpha = 2$ and $D_2 = 1/2m$, Eq. (60) gives the well-known density matrix for a one-dimensional free particle (see Eq. (10–46) of Ref. [3] or Eq. (2–61) of Ref. [21]):

$$\rho^{(0)}(x,\beta|x_0) = \left(\frac{m}{2\pi\hbar^2\beta} \right)^{1/2} \exp\left\{ -\frac{m}{2\hbar^2\beta}(x-x_0)^2 \right\}. \tag{61}$$

The Fourier representation $\rho_L^{(0)}(p,\beta|p')$ of the fractional density matrix $\rho_L^{(0)}(x,\beta|x_0)$, defined by

$$\rho_L^{(0)}(p,\beta|p') = \int_{-\infty}^{\infty} dx\, dx_0\, \rho_L^{(0)}(x,\beta|x_0) \times \exp\left\{ -\frac{i}{\hbar}(px - p'x_0) \right\}$$

can be rewritten as

$$\rho_L^{(0)}(p,\beta|p') = 2\pi\hbar\, \delta(p-p') e^{-\beta D_\alpha |p|^\alpha}.$$

In order to obtain a formula for the fractional partition function in the limit of fractional classical mechanics, let us study the case when $\hbar\beta$ is small. Repeating consider, similar to Feynman's, (see Chap. 10 of Ref. [3]) for the fractional density matrix $\rho_L(x,\beta|x_0)$ we can write the equation

$$\rho_L(x,\beta|x_0) = e^{-\beta V(x_0)} \frac{1}{2\pi\hbar} \times \int_{-\infty}^{\infty} dp \exp\left\{ i\frac{p(x-x_0)}{\hbar} - \beta D_\alpha |p|^\alpha \right\}.$$

Then the partition function in the limit of classical mechanics becomes

$$Z = \int_{-\infty}^{\infty} dx\, \rho_L(x,\beta|x) = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{2\pi\hbar(\beta D_\alpha)^{1/\alpha}} \int_{-\infty}^{\infty} dx\, e^{-\beta V(x)}. \tag{62}$$

This simple form for the partition function is only an approximation, valid if the particles of the system cannot wander very far from their initial positions in the time $\hbar\beta$. The limit on the distance which the particles can wander before the approximation breaks down can be estimated in Eq. (60). We see that if the final point differs from the initial point by as much as

$$\Delta x \simeq \hbar(\beta D_\alpha)^{1/\alpha} = \hbar\left(\frac{D_\alpha}{kT}\right)^{1/\alpha},$$

the exponential function of Eq. (60) becomes greatly reduced. From this, we can infer only intermediate points on paths which do not contribute greatly to the path integral of Eq. (60). If the potential $V(x)$ does not alter very much as x moves over this distance, then the fractional classical statistical mechanics is valid.

The density matrix $\rho_L(x,\beta|x_0)$ obeys the fractional differential equations

$$-\frac{\partial \rho_L(x,\beta|x_0)}{\partial \beta} = -D_\alpha(\hbar\nabla_x)^\alpha \rho_L(x,\beta|x_0) + V(x)\rho_L(x,\beta|x_0) \quad (63)$$

or

$$-\frac{\partial \rho_L(x,\beta|x_0)}{\partial \beta} = H_\alpha \rho_L(x,\beta|x_0), \quad \rho_L(x,0|x_0) = \delta(x-x_0),$$

where the fractional Hamiltonian H_α is defined by Eq. (30). Thus Eqs. (57)–(60) and (63) are the basic equations for our FSM.

VII. CONCLUSION

We have developed a path integral approach to FQM and FSM. The approach is based on functional measures generated by the stochastic process of the Lévy flight whose path fractional dimension is different from the fractional dimension of the Brownian path. As shown by Feynman and Hibbs, the fractality (the difference between topological and fractional dimensions) of the Brownian paths leads to standard (nonfractional) quantum mechanics and statistics. The fractality of the Lévy paths as shown in the present paper leads to fractional quantum mechanics and statistics. Thus we develop a fractional background which leads to fractional (nonstandard) quantum and statistical mechanics.

The Feynman quantum-mechanical and Wiener statistical mechanical path integrals are generalized, and as a result we have fractional quantum-mechanical and fractional statistical mechanical path integrals, respectively. A fractional generalization of the Schrödinger equation has been derived using the definition of the quantum Riesz fractional derivatives. We have defined the fractional Hamilton operator and proved its hermiticity. The relation between the energy and the momentum of a nonrelativistic fractional quantum-mechanical particle has been found. The equation for the fractional plane wave function was obtained. We have derived a free particle quantum-mechanical kernel using Fox's H function. In the particular Gaussian case $(\alpha=2)$, the fractional kernel takes the form of Feynman's well-known kernel. For the Lévy wave packet the position and momentum uncertainties were calculated analytically. The fractional generalization of the Heisenberg uncertainty relation has been established.

Equations (12)–(15), (21)–(24), (28), (30), and (54) are the basic equations for our FQM. Following the general rule and replacing by $it \rightarrow \hbar\beta$, we obtain the path integral formulation of the FSM. An equation for the fractional partition function has been derived, and the fractional quantum-statistical action introduced into the quantum statistical mechanics. The density matrix of a free particle has been expressed analytically in terms of Fox's H function. It is shown that Eq. (60) for the fractional density matrix in a special Gaussian case $(\alpha=2)$ gives the well-known equation for the density matrix of a free particle in one dimension (see Eq. (2–61) of Ref. [21]). We have found the formula for the fractional partition function in the limit of fractional classical mechanics, and discuss the validity of this formula. A fractional differential equation of motion of density matrix has been established. Equations (57)–(60) and (63) are the basic equations for our FSM. We finally mention that the developed fractional path integral approach to quantum and statistical mechanics can easily be generalized to a d-dimensional consideration, using a d-dimensional generalization of the fractional and the Lévy path integral measures.

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